Addendum: some notes on the geometrical concepts used in the proof

In this note we will use examples (and pictures) to examine the geometrical properties of some of the polytopes encountered in the proof, and to examine the relations between these polytopes (and their volumes), in the hope that the underlying geometrical interpretation of the core formulas in the proof becomes more apparent. The pictures in this note were created with GeoGebra 5 beta release.

We start by repeating some definitions and partial results from the proof.

In the proof we studied the parametric polytope $P(S)$ defined by

$$\begin{cases} 0 \leq y_1, y_2, \ldots, y_m \leq 1 \\ y_1 + y_2 + \cdots + y_m \leq S \end{cases}$$

with $S \in \mathbb{R}$ and $0 \leq S \leq m$. As in the proof, let $F(S)$ be the face of $P(S)$ lying in the bounding hyperplane $y_1 + y_2 + \cdots + y_m = S$, and let $F^*(S)$ be its orthogonal projection onto the hyperplane $y_m = 0$.

We proved that the coefficient N_{m-1} of degree $m-1$ of $N(x,m)$ in x (m even) is equal to

$$vol_{m-1}(F^*(m/2))/m!$$

In the proof we also introduced the polytopes (simplexes) defined by

$$\begin{cases} 0 \leq y_1, y_2, \ldots, y_m \\ y_1 + y_2 + \cdots + y_m \leq S \\ \forall k \in I: y[k] \geq 1 \end{cases}$$

where I can be any subset of $[m] = \{1, \ldots, m\}$ (I replaced the I^* from the proof by I here, for ease of notation). Let us call the above simplex $G_S(I)$, and let $G_S^*(I)$ be its orthogonal projection onto the hyperplane $y_m = 0$.

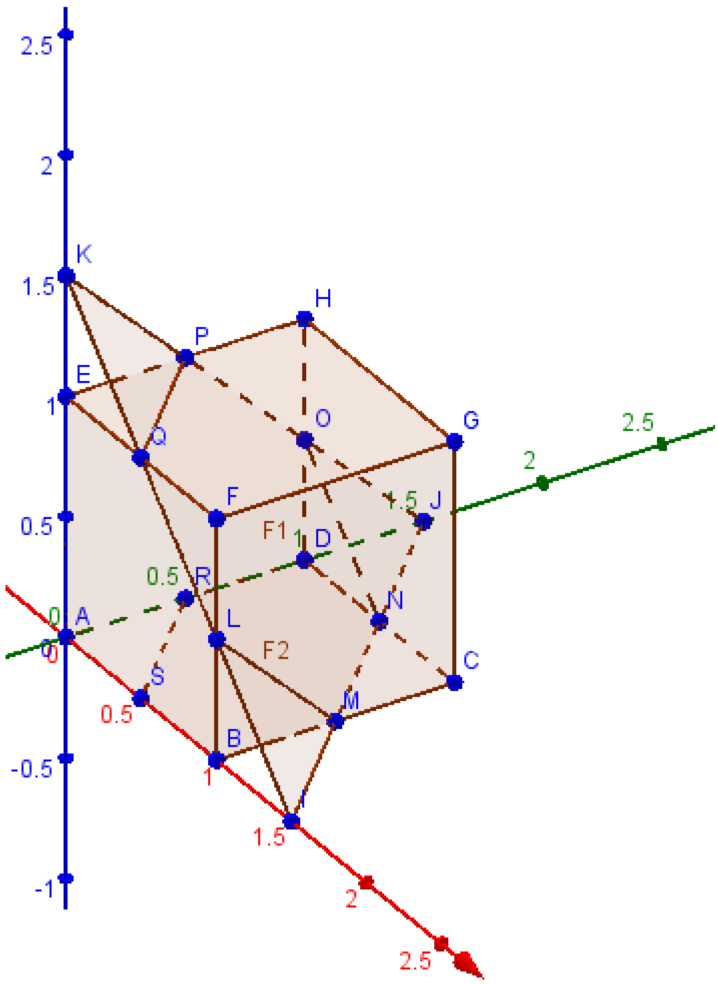
In this note we will examine, by means of some examples, the geometrical properties of the polytopes $P(S)$, $F(S)$, $F^*(S)$, $G_S(I)$, and $G_S^*(I)$, and the relations between these polytopes (and their volumes), in the hope that the underlying geometrical meaning of the following two summation formulas, which play an important role in the proof, becomes more apparent:

$$vol_m(P(S)) = \frac{1}{m!} \sum_{k=0}^{\lfloor S \rfloor} (-1)^k \binom{m}{k} (S-k)^m$$

$$vol_{m-1}(F^*(S)) = \frac{1}{(m-1)!} \sum_{k=0}^{\lfloor S \rfloor} (-1)^k \binom{m}{k} (S-k)^{m-1}$$

As can be seen from the definition, the polytope $F(S)$ is the intersection of the unit hypercube and the hyperplane $y_1 + y_2 + \cdots + y_m = S$, and $P(S)$ is the part of the unit hypercube “under” this hyperplane (the part lying on the side which also contains the origin).

The easiest mental picture can be made for the case $m = 3$ (so we drop all the "hyper" prefixes, vol_m means "volume of", and vol_{m-1} means "area of"). In the proof we assumed m to be even, but for the current discussion this restriction is not necessary. Also, the case $m = 2$ seems to be too trivial as an example, and for $m \geq 4$ it becomes much harder to make clear pictures. With $m = 3$ and $S = \lfloor m/2 \rfloor = 1$ the above summations have only two terms ($k = 0$ and $k = 1$), which is not enough to make the ideas very clear. So in the examples for $m = 3$ we do not fix $S = 1$ in advance (but we still assume that $0 \leq S \leq m$).



The first picture shows the case where $m = 3$ and $S = 3/2$. In the pictures, the coordinate axes are colored in red (y_1), green (y_2) and blue (y_3). We can clearly recognize the unit cube, the plane $y_1 + y_2 + y_3 = S$, and the face $F(S)$, which in this case happens to be hexagonal (F1 in picture; its vertices are the points with labels L, M, N, O, P, and Q). Note that $F(S)$ would have been triangular if $S \leq 1$ or if $S \geq 2$ (see below).

The same picture also shows some of the simplexes (tetrahedrons) $G_S(I)$. All these tetrahedrons have right angles at one of the vertices (the vertex that coincides with one of the vertices of the unit cube, which we will call the *apex* of the tetrahedron, seeing the tetrahedron as a tilted pyramid), and the other three vertices (which we will call *base* vertices) are somewhere on the hyperplane $y_1 + y_2 + y_3 = S$.

The tetrahedrons shown in the picture are $G_S(\phi)$, $G_S(\{1\})$, $G_S(\{2\})$ and $G_S(\{3\})$.

The first one, $G_S(\phi)$, has apex A:(0,0,0), and its three base vertices are I:$(S, 0,0)$, J:$(0, S, 0)$ and K:$(0,0, S)$. Its volume is $S^3/6$.

The second tetrahedron, $G_S(\{1\})$, has apex B:(1,0,0), and its three base vertices are I:$(S, 0,0)$, M:$(1, S-1,0)$ and L:$(1,0, S-1)$. Its volume is $(S-1)^3/6$, because now the three (perpendicular) edges connecting the apex to the base vertices all have the same length $S-1$.

The other two displayed tetrahedrons, $G_S(\{2\})$ and $G_S(\{3\})$, have the same shape and volume as $G_S(\{1\})$. Actually they are just translations of $G_S(\{1\})$, with apex in D:(0,1,0) and E:(0,0,1) respectively. With $m=3$ the unit cube has three vertices where one coordinate is 1 and the other two coordinates are 0, so we have three tetrahedrons with this same shape and volume.

The three tetrahedrons $G_S(\{1\})$, $G_S(\{2\})$ and $G_S(\{3\})$ are included in the larger tetrahedron $G_S(\phi)$. Moreover, because $S=3/2<2$, the three tetrahedrons $G_S(\{1\})$, $G_S(\{2\})$ and $G_S(\{3\})$ are disjoint, and we already know that they have the same volume as $G_S(\{1\})$. It is now clear that

$$vol_3(P(S)) = vol_3(G_S(\phi)) - 3vol_3(G_S(\{1\}))$$

Still for $m=3$ and $S=3/2$, $\lfloor S \rfloor = 1$, and thus the formula for $vol_3(P(S))$ from the proof becomes

$$vol_3(P(S)) = \frac{1}{6}\sum_{k=0}^{1}(-1)^k\binom{3}{k}(S-k)^3 = \binom{3}{0}\frac{S^3}{6} - \binom{3}{1}\frac{(S-1)^3}{6}$$

We see now that both formulas for $vol_3(P(S))$ are in fact one and the same. The first term gives the volume of the tetrahedron $G_S(\phi)$. There is only $1=\binom{3}{0}$ such tetrahedron. The second term gives the total volume of the tetrahedrons $G_S(\{1\})$, $G_S(\{2\})$ and $G_S(\{3\})$. There are $3=\binom{3}{1}$ such tetrahedrons.

The previous line of reasoning can be repeated almost exactly for the orthogonal projections of the tetrahedrons, which then become right-angled equilateral. These are also shown in the same picture, in the plane $y_3=0$. We will also look at this case in detail, although basically this is just a "copy/paste" of what has already been said.

The triangles shown in the picture are $G_S^*(\phi)$, $G_S^*(\{1\})$, $G_S^*(\{2\})$ and $G_S^*(\{3\})$.

All these triangles have right angles at one of the vertices (the apex, the vertex that coincides with one of the vertices of the unit square), and the other two vertices (the base vertices) are somewhere on the line $y_1+y_2=S, y_3=0$.

The first triangle, $G_S^*(\phi)$, has apex A:(0,0,0), and its two base vertices are I:$(S, 0,0)$ and J:$(0, S, 0)$. Its volume is $S^2/2$.

The second triangle, $G_S^*(\{1\})$, has apex B:(1,0,0), and its two base vertices are I:$(S, 0,0)$ and M:$(1, S-1,0)$. Its volume is $(S-1)^2/2$, because now the two (perpendicular) edges connecting the apex to the base vertices all have the same length $S-1$.

The other two displayed triangles, $G_S^*(\{2\})$ and $G_S^*(\{3\})$, have the same shape and area as $G_S^*(\{1\})$. Actually they are just translations of $G_S^*(\{1\})$, with apex in D:(0,1,0) and A:(0,0,0) respectively.

The three triangles $G_S^*(\{1\})$, $G_S^*(\{2\})$ and $G_S^*(\{3\})$ are included in the larger triangle $G_S^*(\phi)$. Moreover, because $S = 3/2 < 2$, the three triangles $G_S^*(\{1\})$, $G_S^*(\{2\})$ and $G_S^*(\{3\})$ are disjoint, and we already know that they have the same area as $G_S^*(\{1\})$. Note that the face $F^*(S)$ is shown as F2 in picture; its vertices are the points with labels B, M, N, D, R, and S. It is now clear that

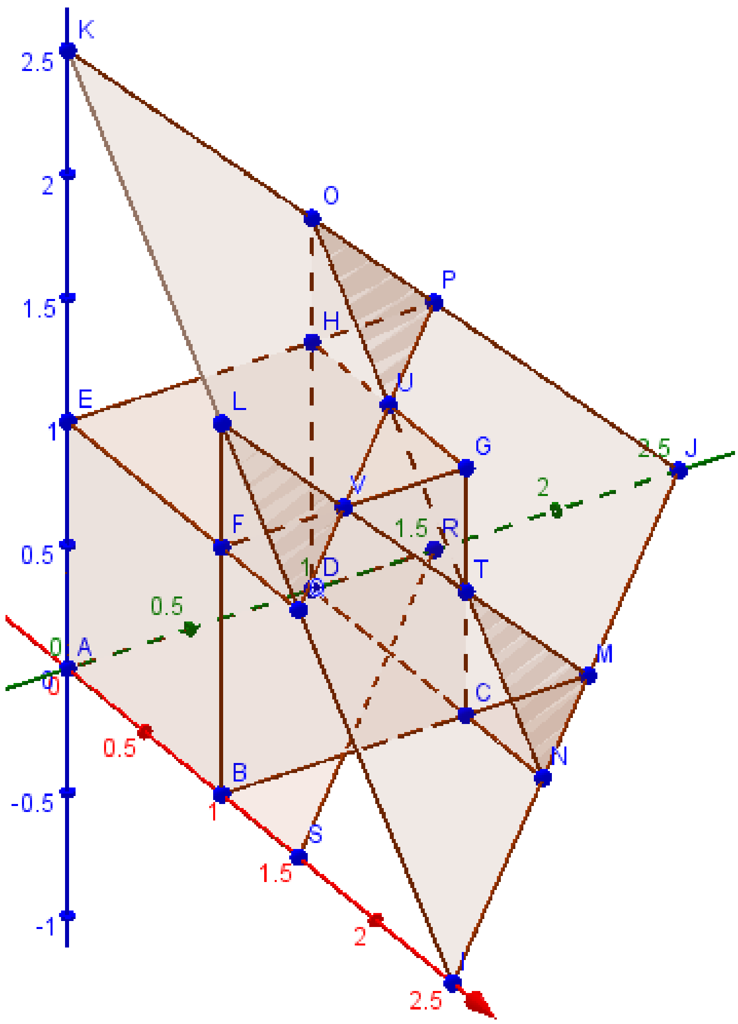
$$vol_2(F^*(S)) = vol_2(G_S^*(\phi)) - 3vol_2(G_S^*(\{1\}))$$

Still for $m = 3$ and $S = 3/2$, $\lfloor S \rfloor = 1$, and thus the formula for $vol_2(F^*(S))$ from the proof becomes

$$vol_2(F^*(S)) = \frac{1}{2}\sum_{k=0}^{1}(-1)^k \binom{3}{k} (S-k)^2 = \binom{3}{0}\frac{S^2}{2} - \binom{3}{1}\frac{(S-1)^2}{2}$$

We see now that both formulas for $vol_2(F^*(S))$ are in fact one and the same. The first term gives the area of the triangle $G_S^*(\phi)$. There is only $1 = \binom{3}{0}$ such triangle. The second term gives the total area of the triangles $G_S^*(\{1\})$, $G_S^*(\{2\})$ and $G_S^*(\{3\})$. There are $3 = \binom{3}{1}$ such triangles.

Next, in order to get three terms, we must take S between 2 and 3. So let us take $S = 5/2$ now. This configuration is depicted in the second picture.



Now $S > 2$, so the face $F(S)$ is triangular. Its vertices are T:$(1,1,S)$, U:$(S,1,1)$ and V:$1,S,1)$. The four tetrahedrons shown in the first picture, $G_S(\phi)$, $G_S(\{1\})$, $G_S(\{2\})$ and $G_S(\{3\})$, are also shown in the second picture (the vertices have the same labels as in the first picture). But this time all the tetrahedrons are larger, and more importantly, the three smaller of these four tetrahedrons, $G_S(\{1\})$, $G_S(\{2\})$ and $G_S(\{3\})$, are *not* disjoint.

The intersection of the tetrahedrons $G_S(\{1\})$ and $G_S(\{2\})$ is the tetrahedron $G_S(\{1,2\})$, and similarly for the other pairs. So now we have an additional level of even smaller tetrahedrons, $G_S(I)$ where $|I| = 2$, which, as we will see, will result in an additional term in the formulas. There are $\binom{3}{2} = 3$ ways to select the two numbers in I from the set $[3]$, so there are indeed 3 additional smaller tetrahedrons: $G_S(\{1,2\})$, $G_S(\{1,3\})$ and $G_S(\{2,3\})$.

All these smaller tetrahedrons also have right angles at one of their vertices (the *apex*, the vertex that coincides with one of the vertices of the unit cube), and the other three vertices (the *base* vertices) are somewhere on the hyperplane $y_1 + y_2 + y_3 = S$.

The tetrahedron, $G_S(\{1,2\})$, has apex C:(1,1,0), and its three base vertices are N:$(S-1,1,0)$, M:$(1,S-1,0)$ and T:$(1,1,S-2)$. Its volume is $(S-2)^3/6$, because now the three (perpendicular) edges connecting the apex to the base vertices all have the same length $S-2$.

The other two displayed tetrahedrons, $G_S(\{1,3\})$ and $G_S(\{2,3\})$, have the same shape and volume as $G_S(\{1,2\})$. Actually, they are just translations of $G_S(\{1,2\})$, with apex in F:(1,0,1) and H:(0,1,1) respectively. With $m = 3$ the unit cube has three vertices where two coordinate are 1 and the other coordinate is 0, so we have three tetrahedrons with this same shape and volume.

The three tetrahedrons $G_S(\{1,2\})$, $G_S(\{1,3\})$ and $G_S(\{2,3\})$ are disjoint ($S < 3$). It is now clear that

$$\begin{aligned} vol_3(P(S)) &= vol_3(G_S(\phi)) - 3(vol_3(G_S(\{1\})) - 3vol_3(G_S(\{1,2\}))) \\ &= vol_3(G_S(\phi)) - 3vol_3(G_S(\{1\})) + 3vol_3(G_S(\{1,2\})) \end{aligned}$$

Still for $m = 3$ and $S = 5/2$ as above, $\lfloor S \rfloor = 2$, and thus the formula for $vol_3(P(S))$ from the proof becomes

$$vol_3(P(S)) = \frac{1}{6}\sum_{k=0}^{2}(-1)^k\binom{3}{k}(S-k)^3 = \binom{3}{0}\frac{S^3}{6} - \binom{3}{1}\frac{(S-1)^3}{6} + \binom{3}{2}\frac{(S-2)^3}{6}$$

We see now, once more, that both formulas for $vol_3(P(S))$ are in fact one and the same. The first term gives the volume of the tetrahedron $G_S(\phi)$. There is only $1 = \binom{3}{0}$ such tetrahedron. The second term gives the total volume of the tetrahedrons $G_S(\{1\})$, $G_S(\{2\})$ and $G_S(\{3\})$. There are $3 = \binom{3}{1}$ such tetrahedrons. The second term gives the total volume of the tetrahedrons $G_S(\{1,2\})$, $G_S(\{1,3\})$ and $G_S(\{2,3\})$. There are $3 = \binom{3}{2}$ such tetrahedrons.

We could again repeat this line of reasoning for the orthogonal projections of the tetrahedrons, as shown in the second picture as well. Since this a completely analogous to what we have done before, no further explanation is necessary.

In general, as the dimension m increases, even more additional levels (i.e., terms) are possible, where each level $k = |I|$ contains $\binom{m}{k}$ simplexes $G_S(I)$ of the same shape and volume. Their apexes coincide with one of the vertices of the unit hypercube having exactly k coordinates equal to 1, and the others equal to 0.

Note that, if we would continue to increase S such that $m - 1 < S < m$ (so $\lfloor S \rfloor = m - 1$), an apex of one of those simplexes $G_S(I)$ would be connected to *each* vertex of the unit hypercube, except to the outer vertex $(1, \dots, 1)$. Since a hypercube of dimension m has exactly 2^m vertices, it must be true that

$$\sum_{k=0}^{m-1} \binom{m}{k} = 2^m - 1$$

which is of course the case. This sum is also equal to the number of proper subsets I of $[m]$.

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